Solutions

3.3: Higher-Order Linear, Homogeneous Equations with Constant Coefficients
Complex Roots

Theorem 3. (Complex Roots)
If the characteristic equation (2) has a complex root $r = a + ib$, then the part of a general solution of the differential equation (1) corresponding to r is of the form

$$e^{ax}(c_1 \cos bx + c_2 \sin bx).$$

Exercise 1. Find the particular solution to the initial value problem

$$y'' - 4y' + 5y = 0, \quad y(0) = 1, y'(0) = 5.$$

$$r^2 - 4r + 5 = 0 \Rightarrow \frac{4 \pm \sqrt{16-20}}{2} = 2 \pm \frac{\sqrt{-4}}{2} = 2 \pm i$$

$$y = e^{2x}(c_1 \cos x + c_2 \sin x) \Rightarrow y(0) = 1 = c_1$$

$$y' = 2e^{2x}(\cos x + c_2 \sin x) + e^{2x}(-\sin x + c_2 \cos x)$$

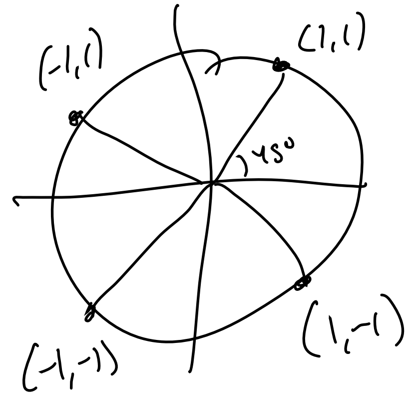
$$\Rightarrow y'(0) = 5 = 2 + c_2 \Rightarrow c_2 = 3$$

So $\quad y = e^{2x}(\cos x + 3\sin x).$

Exercise 2. Find the general solution to

$$y^{(4)} + 4y = 0.$$

$0 = r^4 + 4 = (r^2)^2 + 4 \Rightarrow r^2 = \pm 2i \Rightarrow r =$



or $r = 1 \pm i, -1 \pm i$

So $y = e^{x}(a_1 \cos x + b_1 \sin x) + e^{-x}(a_2 \cos x + b_2 \sin x)$.

Theorem 4. (Repeated Complex Roots)
If the characteristic equation (2) has a repeated complex root $r = a + ib$ of multiplicity k, then the part of a general solution of the differential equation (1) corresponding to r is of the form

$$\sum_{p=0}^{k-1} x^p e^{ax}(c_p \cos bx + d_p \sin bx).$$

Exercise 3. Find the general solution to

$$y^{(6)} - 12y^{(5)} + 63y^{(4)} - 184y^{(3)} + 315y'' - 300y' + 125y = 0$$

which has characteristic equation

$$(r^2 - 4r + 5)^3 = 0.$$

$(r^2 - 4r + 5)$ has roots $2 \pm i$ from Exercise 1.
So these roots now have multiplicity 3.

$y = e^{2x}(a_1 \cos x + b_1 \sin x) + x e^{2x}(a_2 \cos x + b_2 \sin x) + x^2 e^{2x}(a_3 \cos x + b_3 \sin x)$.